Analytical Scaling Investigation of Free Convection Flow of a Hot Fluid Over a Cooler Vertical Isothermal Wall

Olayinka O. Adewumi[1*], Andrew Adebusoye[1], Adetunji Adeniyan[2], Nkem Ogbonna[3], Ayowole A. Oyediran[1]
[1]Department of Mechanical Engineering, University of Lagos, Lagos, Nigeria
[2]Department of Mathematics, University of Lagos, Akoka-Yaba, Lagos, Nigeria
[3]Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Nigeria
E-mail: oadewunmi@unilag.edu.ng; abeti1111@gmail.com ; ayooyediran@hotmail.com

Abstract

Natural convection boundary layer flow of a hot fluid past a cooler semi-infinite plate is analyzed using the method of scale analysis. This kind of flow is a gravity opposing flow. For Prandtl numbers greater than one, there exist two separate boundary layers, namely, the thermal and viscous boundary layers. At moderate Prandtl numbers, these two layers are almost identical but become distinct at higher Prandtl numbers. This is particularly so for heavy oils whose Prandtl number can be as high as 100 000. As the Prandtl number increases, direct numerical integration of the governing equations become inaccurate because there is a very small parameter multiplying the highest derivative. This small parameter is the inverse of the root of the Prandtl number. The method of matched asymptotic expansion is therefore used to compute temperatures, velocities, Nusselt number and skin friction coefficient at moderate and high Prandtl numbers. Five terms inner expansion were matched with five terms outer expansion at the edge of the thermal boundary layer. Results obtained showed that for Prandtl number of up to 100000, both the Nusselt number and skin friction coefficient approached different asymptotes. Also, in all cases considered in this study, the skin friction coefficient and Nusselt numbers for the cold plate immersed in hot fluid are greater than for the case of hot plate immersed in cold fluid when they were compared.

Keywords: *Convection, Scale analysis, Prandtl number, Nusselt number, Boundary layer*

Nomenclature

b	gravitational body force	(m/s^2)
g	dimensionless inner layer stream function	(-)
g'	dimensionless inner layer velocity	(-)
G	dimensionless outer layer stream function	(-)
G'	dimensionless outer layer velocity	(-)
Nu	Nusselt number	(-)
p	pressure	(N/m^2)
Pr	Prandtl number	(-)
Ra	Rayleigh number	(-)
T_w	wall temperature	(˚C)
T_∞	ambient temperature	(˚C)
u, v	velocities in x, y coordinate	(m/s)
x	coordinate normal to wall	(m)
y	coordinate along wall	(m)

Greek symbols

θ	dimensionless inner layer temperature	(-)
$\tilde{\Theta}$	dimensionless outer layer temperature	(-)
η	inner similarity coordinate	(-)
ξ	outer similarity coordinate	(-)
δ	velocity boundary layer	(-)
δ_T	thermal boundary layer	(-)
ϵ	small parameter (Pr$^{-1/2}$)	(-)
ρ	density	(kg/m^3)
ϑ	kinematic viscosity	(m^2/s)
α	thermal diffusivity	(m^2/s

1.0 INTRODUCTION

Many engineering processes in industry make use of heavy oils which are either subjected to heating or cooling. Therefore, it is of great importance that the skin friction on the walls as well as heat transfer rates near and far from the wall is accurately predicted. In a previous study carried out by Adewumi *et al.* [1], an analytical investigation of free convection flow over a heated vertical plate using the Bejan's method of scale analysis [2] was carried out and results obtained were compared with previous works by Kuiken [3], Bachiri and Bovabdallah [4] and Le Fevre [3, 4]. The investigation by Adewumi *et al.* [1] showed that when the difference between Kuiken's scaling method and Bejan's scaling method are taken into consideration for large Prandtl number flows, the results obtained are very similar. Scale analysis has also been used for many unsteady free convection flows [5, 6] and its advantage is that one can easily compare the dominant forces for both low and high Prandtl number flows. The advantage of using scale analysis to obtain the governing boundary layer equations is that two dimensionless parameters "arise" naturally which are the Prandtl and the Rayleigh numbers. The aim of this study is to investigate a class of problems where the temperature of ambient fluid is greater than that at the wall and compare results obtained with the case where the temperature of the wall is greater than that of the fluid [1]. There have been few studies on free convection gravity-opposing boundary-layer flows [7, 8] but none have used the Bejan's method of scale analysis with method of matched asymptotic expansion in investigating heat transfer and skin friction. In this paper, results of temperature profiles, velocity profiles, Nusselt number and shear stress in the inner (thermal) boundary layer for free convection gravity-opposing flows are found and compared with results previously obtained for gravity-aided flows using scale analysis, similarity formulation and the method of matched asymptotic expansion.

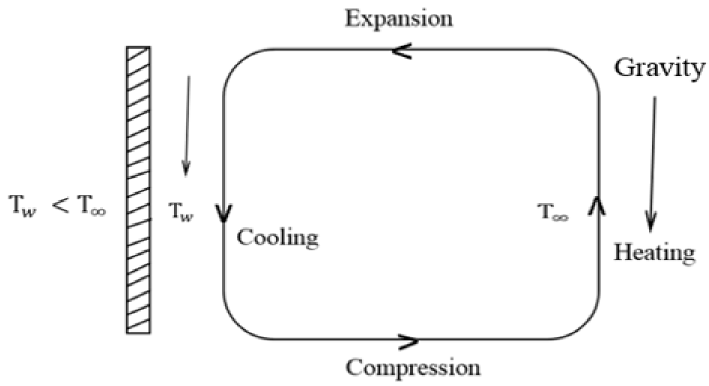

Fig. 1 Gravity opposing fluid flow over a vertical wall

Figure 1 shows natural convection flow of a hot fluid at temperature T_∞ over a cooler wall of height H at temperature T_w. Fluid closest to the wall exchange heat with the wall and causes cooling of the fluid which is denser than the hot fluid further from the wall. A cyclic movement occurs as shown in the wall. The fluid is assumed to be Newtonian and steady flow. Also, all the thermo-physical properties of the fluid are assumed to be constant except density and the Boussineq's approximation was used to obtain equation (2). The flow is known as gravity-opposing flow because buoyancy forces as a result of heating pushes up the flow against gravity.

2.0 Model Formulation

Based on the boundary layer theory, the two-dimensional flow of a hot fluid over a cooler isothermal wall can be mathematically modelled as in Eq. 1

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} - b\beta(T - T_\infty) \qquad (2)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \quad (3)$$

Subject to the boundary conditions in equation (4)

$$\begin{gathered} u=0,\ v=0,\quad \mathrm{T}=\mathrm{T}_w\quad at\ x\to 0 \\ u\to 0,\quad \mathrm{T}=T_\infty\quad at\quad x\to\infty \end{gathered} \quad (4)$$

Where the last term in equation (2) is the usual buoyancy term and it opposes the motion in this study. Dimensionless temperature θ is defined as

$$\theta=\frac{T-T_\infty}{T_w-T_\infty}. \quad (5)$$

3.0 METHODOLOGY

Following Bejan's method of scale analysis and similarity formulation for large Prandtl number flows, we introduce similarity variables in the thermal boundary layer as:

$$\delta_T\sim(Ra_H Pr)^{-\frac{1}{4}},\eta=\frac{x}{\delta_T}\ ,\ \psi_{inner}=\alpha Ra_y{}^{\frac{1}{4}}g(\eta) \quad (5)$$

Similarly, the similarity variables in the velocity layer where inertia balances viscous term are,

$$\delta\sim HPr^{\frac{1}{2}}Ra_H^{-\frac{1}{4}},=\frac{x}{\delta},\psi_{outer}=\vartheta Ra_y^{\frac{1}{4}}Pr^{-\frac{1}{2}}G(\xi) \quad (6)$$

Substituting equations (5) and (6) into equations (2) to (3), we obtain

$$\frac{3}{4}g\theta'=\theta'' \quad (7)$$

$$\frac{1}{Pr}\left[\frac{g'^2}{2}-\frac{3}{4}gg''\right]=-g'''-\theta \quad (8)$$

Which are the inner layer equations subject to the boundary conditions as shown in equation (9).

$$g=0,\quad g'=0,\ \theta=1\ \ at\ \eta=0 \quad (9)$$

While the outer layer equations are,

$$\frac{3}{4}G\theta'=\frac{1}{Pr}\widetilde{\Theta}'' \quad (10)$$

$$\frac{1}{Pr}\left[\frac{G'^2}{2}-\frac{3}{4}GG''\right]=-\frac{1}{Pr}G'''-\widetilde{\Theta} \quad (11)$$

Subject to the following boundary conditions,

$$G'=0,\ \widetilde{\Theta}=0\ \ as\ \ \xi\to\infty \quad (12)$$

For moderate Prandtl numbers, both inner and outer layers overlap and consequently, $(g\equiv G)$, $(\theta=\widetilde{\Theta})$

For the case of large Prandtl numbers where the inner layer is quite distinct from the outer layer, the method of matched asymptotic expansion is employed in this study.
Taking $\lim_{\eta\to\infty} \epsilon g = \lim_{\xi\to 0} G$ we have,

$$\lim_{\eta\to\infty} 0 + \epsilon[g_0(\eta) + \epsilon g_1(\eta) + \epsilon^2 g_2(\eta) + \epsilon^3 g_3(\eta)] = \lim_{\xi\to 0} G_0(\xi) + \epsilon G_1(\xi) + \epsilon^2 G_2(\xi) + \epsilon^3 G_3(\xi) + \epsilon^4 G_4(\xi) \quad (13)$$

$$\left.\begin{aligned} G_0(\xi) &= G_{00} + G_{01}\xi + G_{02}\xi^2 + G_{03}\xi^3 + G_{04}\xi^4 + \cdots \\ G_1(\xi) &= G_{10} + G_{11}\xi + G_{12}\xi^2 + G_{13}\xi^3 + \cdots \\ G_2(\xi) &= G_{20} + G_{21}\xi + G_{22}\xi^2 + \cdots \\ G_3(\xi) &= G_{30} + G_{31}\xi + \cdots \\ G_4(\xi) &= G_{40} + \cdots \end{aligned}\right\} \quad (14)$$

$$\left.\begin{aligned} g_0(\eta) &= g_{00} + g_{01}\eta + g_{02}\eta^2 \\ g_1(\eta) &= g_{10} + g_{11}\eta + g_{12}\eta^2 \\ g_2(\eta) &= g_{20} + g_{21}\eta + g_{22}\eta^2 + g_{23}\eta^3 \\ g_3(\eta) &= g_{30} + g_{31}\eta + g_{32}\eta^2 + g_{33}\eta^3 + g_{34}\eta^4 + \cdots \end{aligned}\right\} \quad (15)$$

Matching gives [5],

O (1): $G_{00} = 0 \quad or \quad G_0(0) = 0$

O (ϵ): $g_{02} = 0, \quad g_{01} = G_{01}, \quad g_{00} = G_{10}$

O (ϵ^2): $g_{12} = G_{02}, \quad g_{11} = G_{11}, \quad g_{10} = G_{20}$

O (ϵ^3): $g_{23} = G_{03}, \quad g_{22} = G_{12}, \quad g_{21} = G_{21}, \quad g_{20} = G_{30}$

O (ϵ^4): $g_{34} = G_{04}, \quad g_{33} = G_{13}, \quad g_{32} = G_{22}, \quad g_{31} = G_{31}, \quad g_{30} = G_{40}$

These quantities become the initial conditions for G and boundary conditions for g. We also obtain the O(1), O(ϵ), O(ϵ^2) and O(ϵ^3) equations with all the missing boundary conditions.

4.0 RESULTS AND DISCUSSION

In this section, results obtained in this study where the temperature of the fluid is greater than that of the wall ($T_w < T_\infty$) will be compared with results obtained from previous work done for the case where the temperature of the wall was greater than that of the fluid ($T_w > T_\infty$) [1].

4.1 Temperature profiles and Nusselt number

Figure 2a shows the effect of Prandtl number on the temperature of the fluid moving over a cooler wall. The results in the figure show that temperature increases as Prandtl number increases until it reaches an asymptote. When the temperature profiles of the hot fluid flowing over a cooler wall is compared with temperature results obtained for flow of cold fluid over a heated wall [1] as shown in Figure 2b, we see that for this case the temperature decreases as Prandtl number increases until it reaches an asymptote. This is because Figure 2a shows what happens in a case of gravity-opposing flows while figure 2b shows the case of gravity-aided flows. Also, the thermal boundary layer thickness, δ_T, in Figure 2a is approximately 3 for the case of hot fluid over a cooler wall while for the case of cold fluid over a heated vertical wall δ_T is approximately 6 (as seen in Figure 2b). This comparison shows that for this case of gravity-aided natural convection flow, the thermal boundary layer is about 50% thicker than that of gravity-opposing flow.

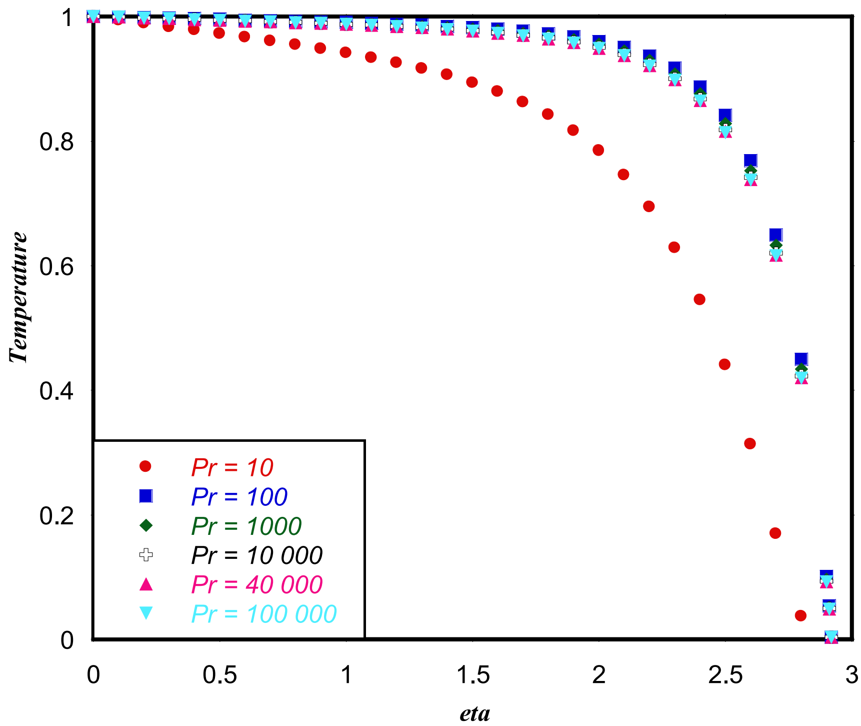

Figure 2(a) Similarity temperature profiles hot fluid over a cooler wall

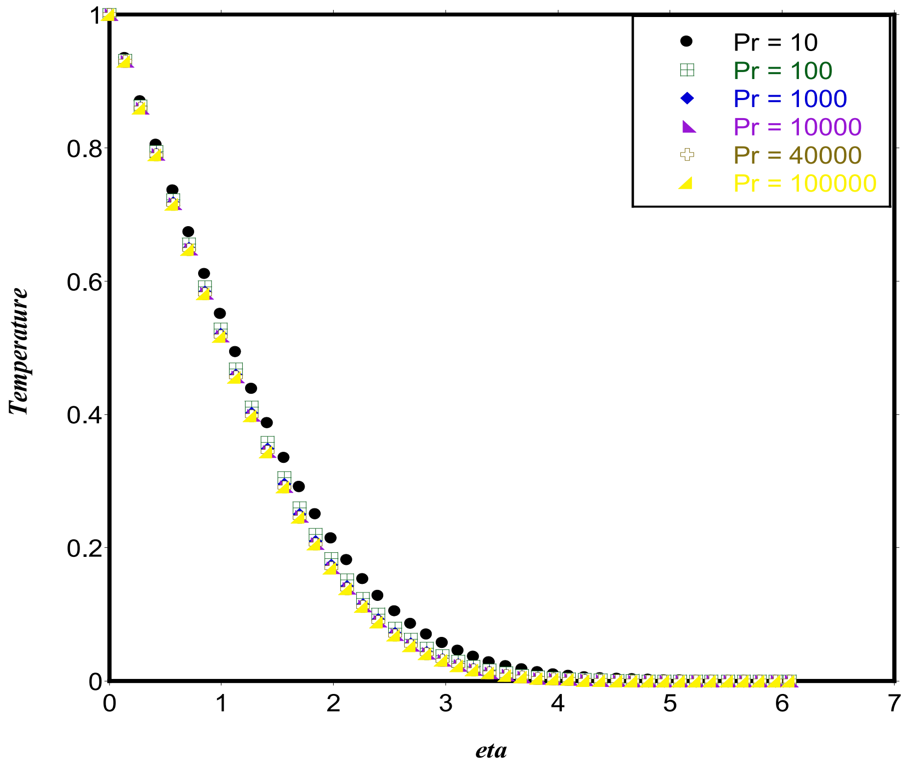

Figure 2(b) Similarity temperature profiles cold fluid over a heated wall

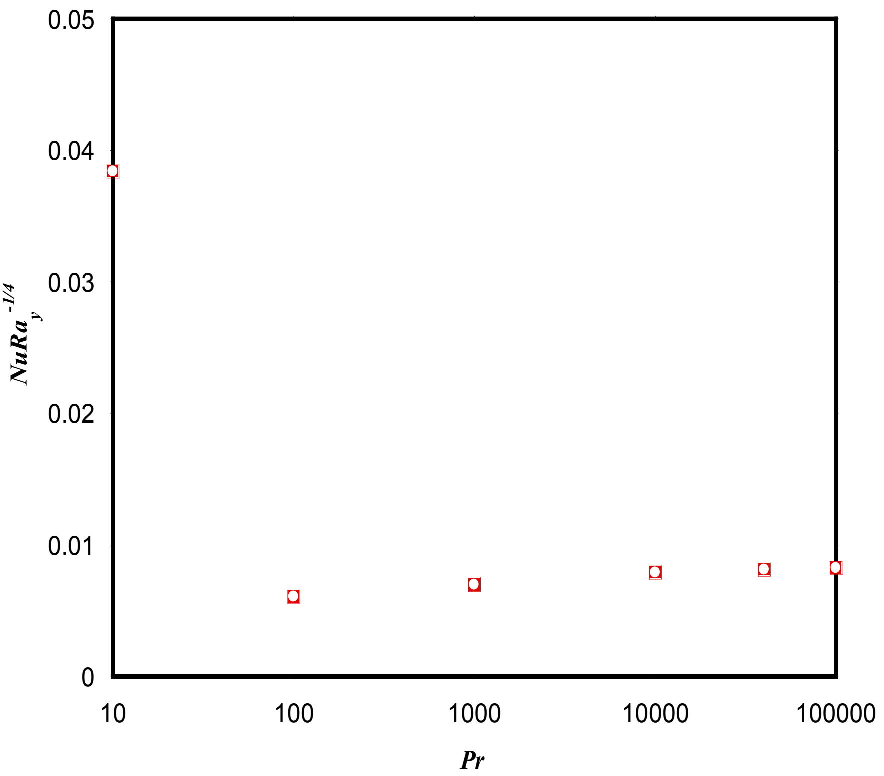

Figure 3(a) Effect of increase in Prandtl number on $NuRa_y^{-1/4}$ for a hot fluid over a cooler wall

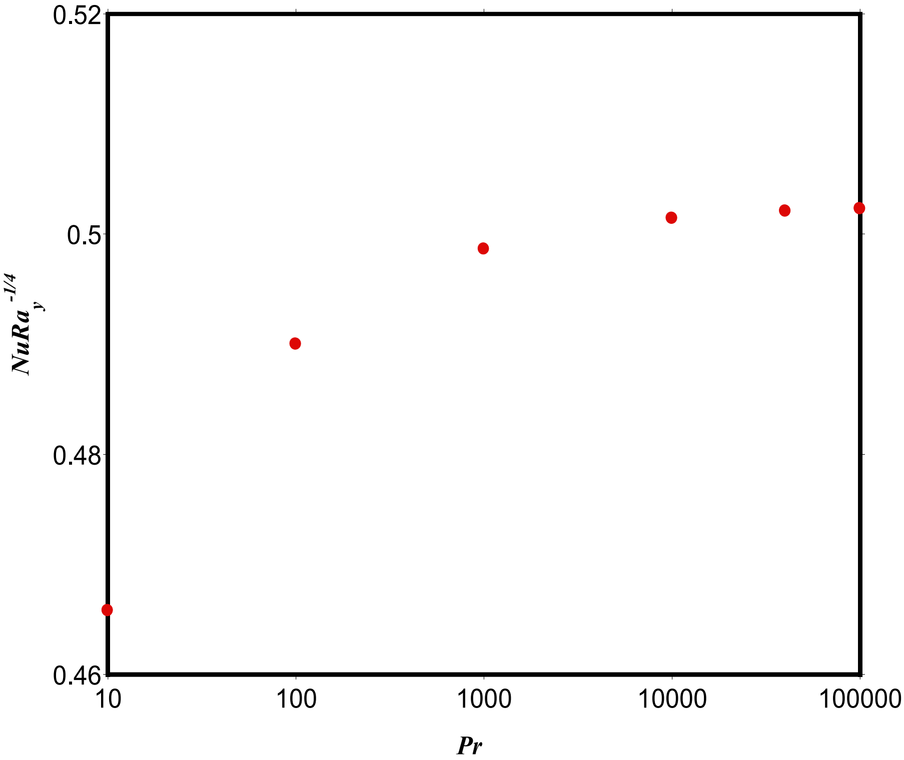

Figure 3(b) Effect of increase in Prandtl number on $NuRa_y^{-1/4}$ for a cold fluid over a heated wall

Table 1: Comparison between $NuRa_y^{-1/4}$ for gravity-opposing and gravity-aided flow

Pr	$NuRa_y^{-1/4}$ (gravity-opposing flow)	$NuRa_y^{-1/4}$ (gravity-aided flow)	*% Difference*
10	0.03839	0.46581	91.76
100	0.00605	0.49004	98.77
1 000	0.00696	0.49862	98.60
10 000	0.00788	0.50143	98.43
40 000	0.00813	0.50209	98.38
100 000	0.00822	0.50231	98.36

Figure 3 and Table 1 shows results of Nusselt number as Prandtl number increases for both gravity-opposing and gravity-aided flows. The Nusselt number is calculated using the expression in equation (16) below [1].

$$\frac{Nu}{Ra_y^{1/4}} = -2^{-1/2}\left(\theta_0'|_{\eta=0} + \epsilon\theta_1'|_{\eta=0} + \epsilon^2\theta_2'|_{\eta=0} + \epsilon^3\theta_3'|_{\eta=0}\right) \tag{16}$$

The product of Nusselt number and inverse of the Rayleigh number raised to a quarter plots for Prandtl number range of $10 \leq Pr \leq 100{,}000$ for this study is shown in Figure 3a. There is a sharp decrease in $NuRa_y^{-1/4}$ as the Prandtl number increases from 10 to 100. Thereafter, there is a steady increase in $NuRa_y^{-1/4}$ until it reaches an asymptotic value of approximately 0.008 as Prandtl number increases from 100 to 100,000. This shows that for a hot fluid flowing over a cooler vertical wall, the heat transfer rate for large Prandtl number fluids is very low. When these results are compared with previous results obtained for a cold fluid flowing over a heated vertical wall [1] as shown in Figure

3b, we see that as the Prandtl number increases, the $NuRa_y^{-1/4}$ also increases steadily and approaches an asymptotic value of 0.502. We can conclude from these results that gravity-opposing flows cause a reduction of about 99% in the rate of heat transfer when compared to gravity-aided flows.

4.2 Velocity profiles and shear stress

Figure 4a shows velocity profiles for this present study when hot fluid flows over a cooler vertical wall for different Prandtl numbers. The negative value shows that the fluid is moving in the direction of gravity close to the wall surface as shown in Figure 1. For Prandtl number of 10, the effect of the gravity-opposing flow is negligible and as a result the fluid moves in an upward direction similar to the gravity-aided flow. As the Prandtl number increases from 100 to 100,000, the velocity of the fluid increases downward until it reaches an asymptote. Figure 4b shows results of velocities for a gravity-aided flow previously obtained [1]. For this case, the velocity of the fluid increases upward as Prandtl number increases until it reaches an asymptote. This difference in trend is expected because one case is a gravity-opposing flow while the other is a gravity-aided flow.

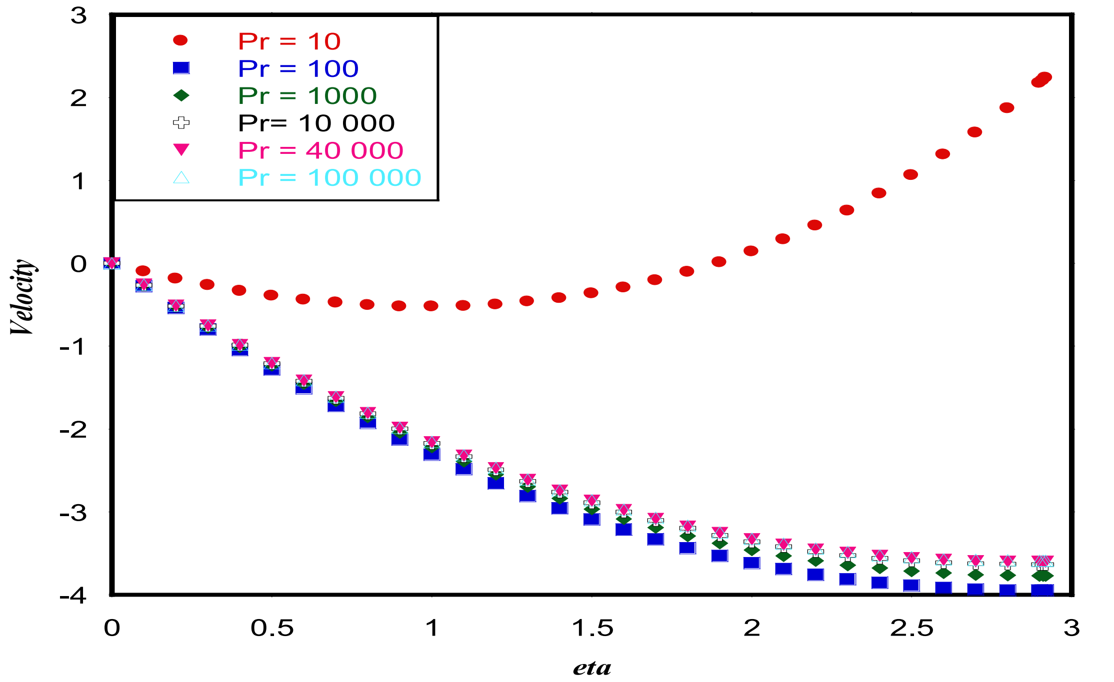

Figure 4(a) Similarity velocity profiles (approximate analytical results) for a hot fluid over a cooler wall

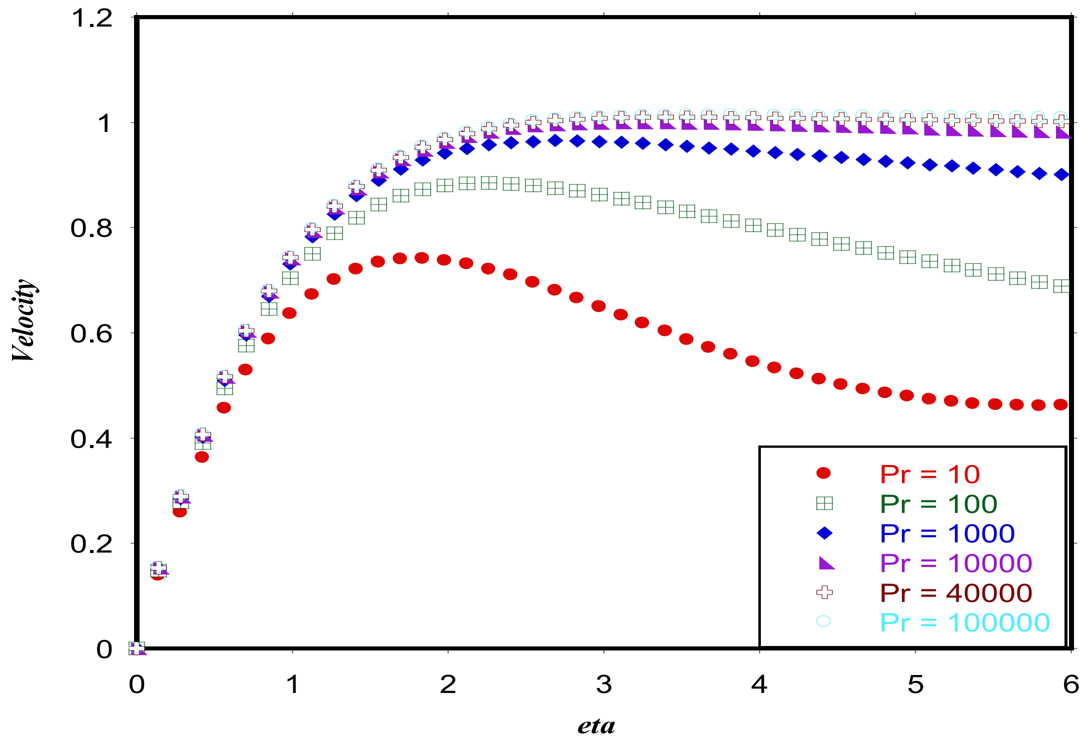

Figure 4(b) Similarity velocity profiles (approximate analytical results) for a cold fluid over a heated wall

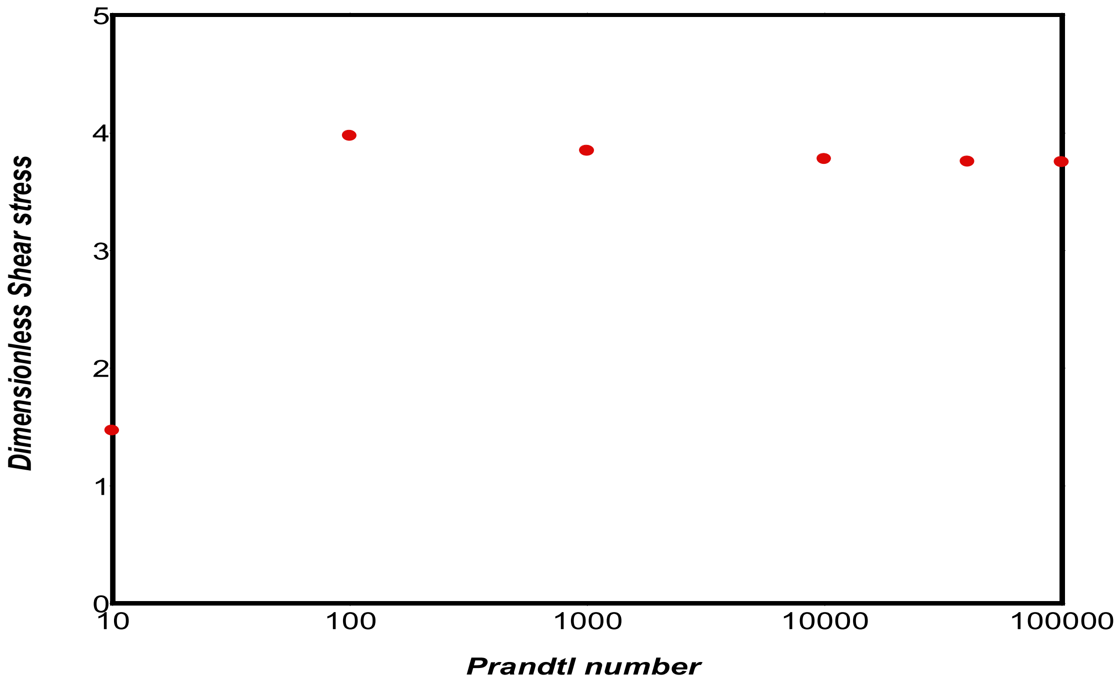

Figure 5(a) Effect of increase in Prandtl number on dimensionless shear stress for a hot fluid over a cooler wall

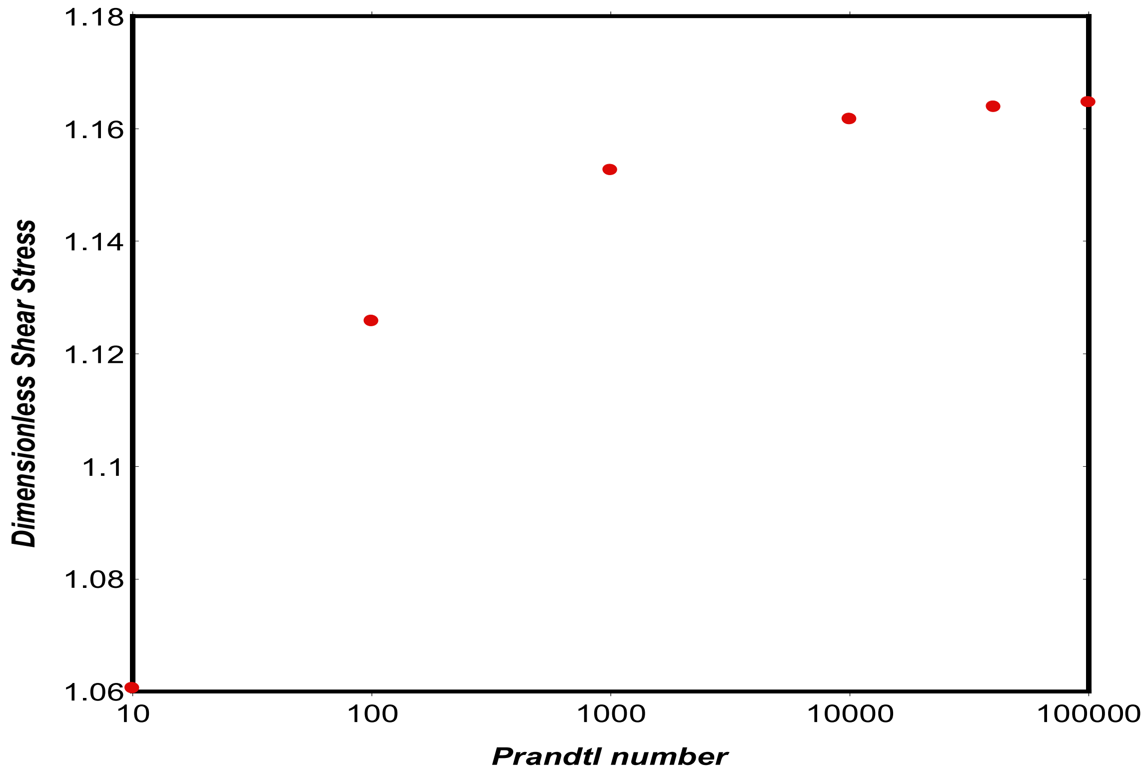

Figure 5(b) Effect of increase in Prandtl number on dimensionless shear stress for a cold fluid over a heated wall

Table 2: Comparison between dimensionless shear stress for gravity-opposing and gravity-aided flow

Pr	(gravity-opposing flow)	(gravity-aided flow)	*% Difference*
10	1.4692	1.0605	27.82
100	3.9732	1.1258	71.67
1 000	3.8484	1.1526	70.04
10 000	3.7747	1.1617	69.22
40 000	3.7566	1.1639	69.01
100 000	3.7498	1.1647	68.94

Results of dimensionless shear stress due to frictional effects of the vertical wall on the fluid are shown in Figure 5 and Table 2. The expression for dimensionless shear stress is given in equation (17) below [1].

$$g''(0) = \tau_0 \frac{y^2}{\sqrt{2}\mu_0 \alpha Ra_y^{3/4}} \tag{17}$$

In Figure 5a, we see that the shear stress at *Pr* = 10 is very low compared to the shear stress value at higher Prandtl numbers. As Prandtl number increases from 10 to 100, there is a sharp increase in the value of the shear stress. Thereafter, as the Prandtl number increases from 100 to 100,000, the shear stress decreases until it reaches an asymptotic value of ~ 3.75. When we compare results of shear stress for gravity-opposing and gravity-aided flow in Figure 5b [1], we see that shear stresses are lower for flow of cold fluid over a heated wall. For this case, shear stress value increases as Prandtl number increases until it reaches an asymptotic value of ~1.16.

5.0 CONCLUSION

This paper presented investigations into the free convection flow of hot fluid over a cooler wall using numerical method and Bejan's method of scale analysis with method of matched asymptotic expansion. These methods were used to determine the inner layer velocity, temperature, Nusselt number and shear stress for Prandtl numbers of 10 to 100,000. Results obtained were also compared with previous results obtained for free convection flow of cold fluid over a heated wall and the following conclusions were reached;

For the case of hot fluid flowing over a cooler wall, $NuRa_y^{-1/4}$ approaches an asymptotic value of ~ 0.008 which is much lower than the value previously obtained for cold fluid flowing over a heated wall. Also, the shear stress decreases as Prandtl number increases from 100 to 100,000 until it reaches an asymptotic value of ~3.75. The reverse was observed for cold fluid flowing over a heated wall where shear stress increases with increase in Prandtl number until it reached an asymptotic value of ~ 1.16.

REFERENCES

Adewumi, O.O., Adebusoye, A., Adeniyan, A., Ogbonna, N. and Oyediran, A.A., (2017)., Scale analysis and asymptotic solution for natural convection over a heated flat plate at high Prandtl numbers, Proc. of Constructal & Second law Conf., pp. 242-256 .

Bejan, A., (2013), Convection Heat Transfer, John Wiley & Sons, pp. 168-232.

Kuiken, H.K., (1968), An asymptotic solution for large Prandtl number free convection, J. Engr. Math., II(4), pp. 356-371.

Bachiri, M. and Bouabdallh, A., (2010), An analytical investigation of the steady-state natural convection boundary layer flow on a vertical plate for a wide range of Prandtl numbers, Heat Transf. Engr., 31(7), pp. 608-616.

Armfield, S.W., Patterson, J.C., Lin, W., (2007), Scaling investigation of the natural convective boundary layer flow on an evenly heated plate, Int. J. Heat Mass Transfer, 50, pp. 1592-1602.

Patterson, J.C., Imberger, J., Unsteady natural convection in a rectangular cavity, Journal of Fluid Mechanics, 100, 65-86, (1980).

Mongruel, A., Cloitre, M. and Allain, C., (1996), Scaling of boundary layer flows driven by double diffusive convection, Int. J. Heat Mass Transf., 39(18), pp. 3899-3910.

Ishak, A., Nazar, R., Arifin, N.M. and Pop, I., (2008), Dual solutions in mixed convection flow near a stagnation point on a vertical porous plate, Int. J. Thermal Sci., 47(4), pp. 417-422.