Analytical Solutions of Thermal-Mechanical Vibration Models of Pinned-Pinned Fluid-Conveying Single-Walled Carbon Nanotubes Resting on Elastic Foundation

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Abstract
Pinned-pinned single-walled carbon nanotubes (SWCNTs) have attracted a lot of interest in recent years due to their suitability for a wide range of applications, such as field emission and vacuum microelectronic devices, nanosensors, and nanoactuators. Based on two simply supported beam-bending models and mode analysis, analytical solutions are developed in the present study to deal with the resonant frequency of a SWCNT. The resonant frequency shift of the pinned-pinned SWCNTs caused by change in temperature and interaction with both a Winkler and a Pasternak elastic medium are examined in order to explore the suitability of SWCNTs as a cooling device and resonators in quantum computer designs. The simulation results reveal that the increase in temperature and the non-local parameter decreases the resonant frequency. In contrast, the resonant frequency increases with increase in the stiffness of the elastic medium except that Tai-Ping Chang model is insensitive to changes in the Pasternak constant. Furthermore, the modified Haw-Long Lee model predicts a larger flutter compared to the modified Tai-Ping Chang et al. model. Therefore, the modified Tai-Ping Chang model is better used to analyze the thermal-mechanical vibration of a SWCNT.

Keywords: *Euler-Bernoulli, nanotubes, Pasternak-type, Winkler elastic constant*

1.0 INTRODUCTION

Carbon nanotubes (CNTs) have been known to have remarkable mechanical and physical properties leading to many potential applications (Dresselhaus *et al*., 2004). The mechanical vibration properties of CNTs have been investigated using experimental techniques (Yakobson and Avouris, 2001; Yao *et al*., 2008) and theoretical analytical techniques (Natsuki *et al*., 2005). There are two categories of theoretical analysis techniques for studying the mechanical behaviour of CNTs. First is the atomic modeling techniques and second is the continuum-based techniques (such as the elastic beam and the elastic shell theories) (Liew *et al*., 2004). Many studies related to the field are depicted in the references (Yoon *et al*., 2006; Natsuki *et al*., 2004; Wang *et al*., 2006; and Zhang *et al*., 2005). The non-local elasticity theory was first initiated by Eringen (2002). The importance of nonlocal elasticity theory stimulated the researchers to investigate the properties of the micro/nanostructures more accurately and conveniently. Application of nonlocal continuum theory to nanotechnology was initially reported by Peddieson *et al*. (2003).

The non-local elasticity theory takes into consideration the nanoscale effects. It assumes that the stress at a point is a function of the strain at every point in the material domain. The effect of thermal changes on the frequencies of the nanotubes has been the focus of recent research works. Using a non-local Timoshenko beam theory; Benzair *et al*. (2008) investigated the effect of temperature changes on the free vibration of single-walled carbon nanotubes. Wang *et al*. (2008) studied thermal effects on free vibration of fluid conveying single-walled carbon nanotubes.

In this paper, the non-local Euler-Bernoulli elastic beam theory is used to investigate the vibrational behavior of a simply supported single-walled carbon nanotubes (SWCNTs) embedded in a two-parameter elastic medium. Both Winkler-type and Pasternak-type models are employed to simulate the interaction of the SWNTs with a surrounding elastic medium more accurately. In addition, the influences of non-local effects and temperature changes on frequency response of the model are examined.

2.0 METHODOLOGY

2.1. Theoretical Framework

The models of Chang *et al*. (2011) and Haw-Long Lee *et al.* (2009) are considered for this work, but, taking into consideration the effects of the vibration characteristics.

2.1.1 Model 1: Modified Tai-Ping Chang and Mei-Feng Liu model

The model equations of Chang *et al*. (2011) have been modified to take into consideration the effects of temperature, non-local parameter and the two elastic foundation parameters.

Based on the non-local elasticity theory, the thermal-mechanical behaviour of a SWCNT conveying fluid can be mathematically modeled as in Eq. 1.

$$\left(EI + (e_0a)^2\theta - (e_0a)^2 m_f V^2\right)\frac{d^4y}{dx^4} - \left(2(e_0a)^2 m_f V\right)\frac{d^4y}{dx^3dt} - ((e_0a)^2 M)\frac{d^4y}{dx^2dt^2} + \left(m_f V^2 - \theta - (e_0a)^2 k_1\right)\frac{d^2y}{dx^2} + \left(2m_f V\right)\frac{d^2y}{dxdt} + (M)\frac{d^2y}{dt^2} + k_2 y = 0 \quad (1)$$

where, term (e_0a) accounts for the small size effects. From thermal elasticity mechanics, force θ due to thermal moments can be written as in Eq. 2 (Chen, 1987):

$$\theta = -\frac{EA}{1-2v} \propto_x T \quad (2)$$

Introducing the following dimensionless parameters:

$$y = \bar{y} * L\ ,\ x = \bar{x} * L\ ,\ t = \bar{t} * T\ ,\ e_0a = \overline{e_n} * L,\ v = \bar{v} * V_c\ ,\ \bar{k}_1 = \frac{k_1L^4}{E\,I}\ ,\bar{k}_2 = \frac{k_2L^4}{E\,I}\ ,\bar{\theta} = \frac{\theta L^2}{E\,I},\beta = \frac{m_f}{M} \quad (3)$$

Substituting Eq. 3 into Eq. 1, a dimensionless form of the governing equation and boundary conditions for model 1 can be obtained as follows in Eqs. 4-5:

$$(1 - (\overline{e_n})^2 * \bar{\theta} - (\overline{e_n})^2 * \bar{v}^2) * \frac{d^4\bar{y}}{d\bar{x}^4} - (\,2(\overline{e_n})^2\beta\,)\frac{d^4\bar{y}}{d\bar{x}^3d\bar{t}} - (\overline{e_n})^2\frac{d^4\bar{y}}{d\bar{x}^2d\bar{t}^2} + \left(\bar{V}^2 - \bar{\theta} - (\overline{e_n})^2\overline{k_1}\right)\frac{d^2\bar{y}}{d\bar{x}^2} + \left(2\sqrt{\beta}\bar{V}\right)\frac{d^2\bar{y}}{d\bar{x}d\bar{t}} + \frac{d^2\bar{y}}{d\bar{t}^2} + \overline{k_2}\bar{y} = 0 \quad (4)$$

$$\bar{y}(0) = \frac{d^2\bar{y}}{dx^2}(0) = \bar{y}(1) = \frac{d^2\bar{y}}{dx^2}(1) = 0 \quad (5)$$

2.1.2 Model 2: Modified Haw-Long Lee and Win-Jin Chang model

The model equations of Haw-Long Lee *et al.* (2009) have been modified to take into consideration the effects of temperature, non-local parameter and the two elastic foundation parameters, thus can be expressed as in Eq. 6:

$$(EI)\frac{d^4y}{dx^4} - ((e_0a)^2M\,)\frac{d^4y}{dx^2dt^2} + \left(m_fV^2 - N_t - (e_0a)^2k_1\right)\frac{d^2y}{dx^2} + \left(2m_fV\,\right)\frac{d^2y}{dxdt} + (M\,)\frac{d^2y}{dt^2} + k_2y = 0 \quad (6)$$

The corresponding boundary conditions for pinned-pinned are expressed in Eq. 7.

$$y(0) = \frac{d^2y}{dx^2}(0) = y(L) = \frac{d^2y}{dx^2}(L) = 0 \quad (7)$$

Introducing the following dimensionless parameters:

$$y = \bar{y}\,L \ , \ x = \bar{x}\,L \ , \ t = \bar{t}\,T \ , \ e_0 a = \overline{e_n}\,L, \ v = \bar{v}\,V_c \ , \ \bar{k}_1 = \frac{k_1 L^4}{E\,I} \ , \ \bar{k}_2 = \frac{k_2 L^4}{E\,I} \ , \bar{\theta} = \frac{N_t L^2}{E\,I} \ , \beta = \frac{m_f}{M} \qquad (8)$$

Substituting Eq. 8 into Eq. 6, a dimensionless form of the governing equation and boundary conditions for model 1 can be obtained as in Eq. 9:

$$(1-(\overline{e_n})^2\overline{N_t})\frac{d^4\bar{y}}{d\bar{x}^4} - (\overline{e_n})^2\frac{d^4\bar{y}}{d\bar{x}^2 d\bar{t}^2} + \left(\bar{V}^2 - \ \bar{\theta} \ - (\overline{e_n})^2\overline{k_1}\right)\frac{d^2\bar{y}}{d\bar{x}^2} + \left(2\sqrt{\beta}\bar{V}\right)\frac{d^2\bar{y}}{d\bar{x}d\bar{t}} + \frac{d^2\bar{y}}{d\bar{t}^2} + \overline{k_2}\bar{y} = 0 \qquad (9)$$

2.2 Method

The models will be analyzed by carrying out eigenvalue analysis to determine the natural frequencies.

2.2.1 Eigenvalue analysis for natural frequencies

The natural frequencies of the models can be obtained analytically following Blevins (2001), and using half-range Fourier series as shown in the unpublished notes by Oyediran (2014) (detail is presented in appendix).

For Eq. 4 and Eq. 9, assuming a solution of the form:

$$y = \bar{y} = a_n\,Sin\left(\frac{n\pi x}{l}\right)\,Sin(wt) + \ a_n\,Sin\left(\frac{n\pi x}{l}\right)\,Cos(wt) \qquad (10)$$

Substituting Eq. 10 into Eq. 4 and Eq. 9 and then the equations can be recast in the form:

$$[A] = [K] - Mw_j[I]\{a\} = 0 \qquad (11)$$

Where $\{a\}$= the vector of the mode shape, w_j = the frequency of vibration

Elements for the Model 1 are in Eq. 12:

$$A_{11} = \ 1 + \bar{e}\left(\bar{\theta} - \ \mathrm{a} + \ \bar{k}_1 - \ x\ \right) - \ (a + \ \bar{\theta}\ ) + \ \bar{k}_2 - \ x \qquad (12a)$$

$$A_{12} = \ \frac{16\sqrt{a\,x\,\beta}}{(3*\pi)} \qquad (12b)$$

$$A_{21} = \ \frac{16\sqrt{a\,x\,\beta}}{(3*\pi)} \qquad (12c)$$

$$A_{22} = \ 16 + 4e\left(4\bar{\theta} - \ 4\mathrm{a} + \ \ \bar{k}_1 - \ x\ \right) - \ (a + \ \bar{\theta}\ ) + \ \bar{k}_2 - \ x \qquad (12d)$$

Elements for the Model 2 are in Eq. 13:

$$A_{11} = \ 1 + \left(\bar{\theta} - \ \mathrm{a} + \ \ \bar{k}_1 - \ x\ \right) \ + \ \ \bar{k}_2 - \ x \qquad (13a)$$

$$A_{12} = \frac{16\sqrt{a\,x\,\beta}}{(3*\pi)} \qquad (13b)$$

$$A_{21} = \ \frac{16\sqrt{a\,x\,\beta}}{(3*\pi)} \qquad (13c)$$

$$A_{22} = 16 + 4 * \left(\bar{\theta} \ + \ \mathrm{a} - \ \ \bar{k}_1 - \ \bar{e}^2 * x\ \right) + \ \bar{k}_2 - \ x \qquad (13d)$$

Setting the determinant of the matrix [A] equal to zero, the dimensionless natural frequencies can be obtained.

3.0 RESULTS AND DISCUSSION

The results of the influence of different parameters on the stability of the buckled solution of the systems were examined. Here flow velocity, *v*, is used as the independent varied parameter.

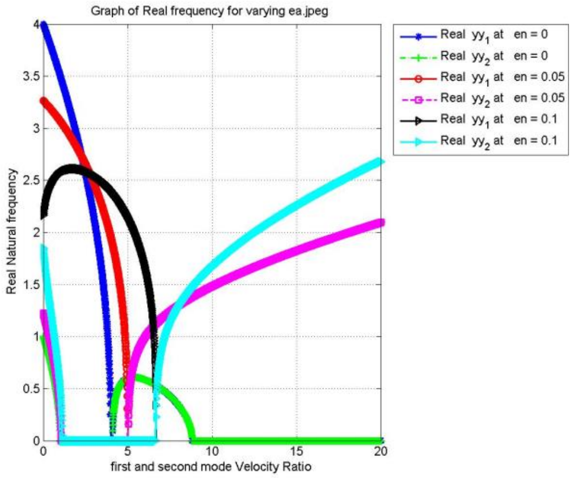

Figure 1(a): Effect of non-local parameter on real frequency for Modified Tai-Ping Chang model

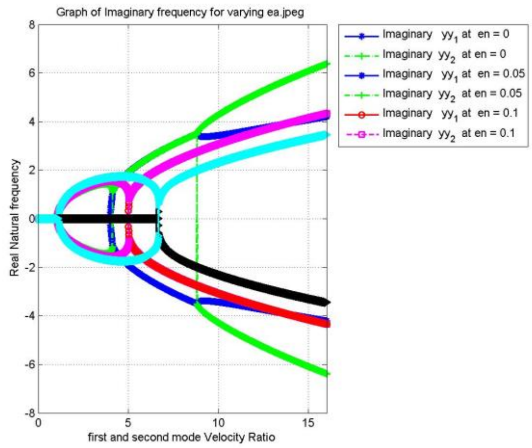

Figure 1(b): Effect of non-local parameter on imaginary frequency for Modified Tai-Ping Chang model

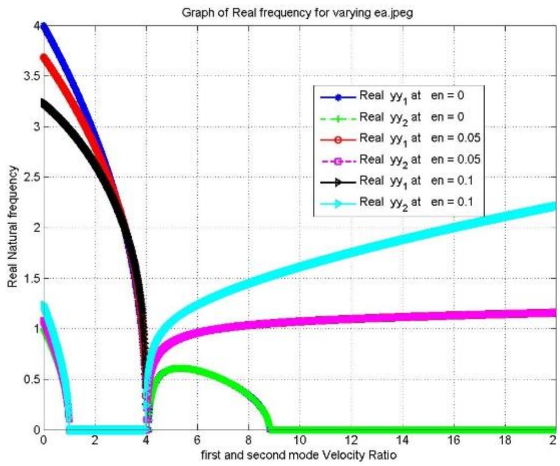

Figure 2(a): Effect of non-local parameter on real frequency for Modified Haw-Long Lee model

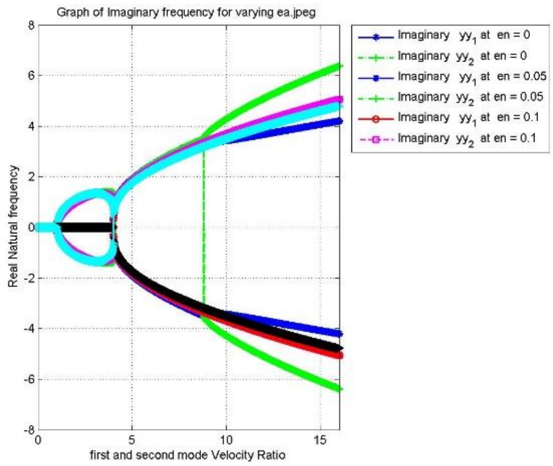

Figure 2(b): Effect of non-local parameter on imaginary frequency for Modified Haw-Long Lee model

The first mode buckling velocity remained constant at a value of 1 while the second mode buckling velocity reduced with increase in e_0a in for the modified Tai-Ping Chang model. **Figures 1 and 2** of simulation for the modified Tai-Ping Chang model carried out at $k_2 = 0$, $k_1 = 0$, $\beta = 0.1$, $\bar{\theta} = 0$, and e_0a was increased from 0 to 0.1. However, for the Haw-Long Lee *et al.* (2009) model both the first mode and the second mode buckling velocities remained constant despite increase in e_0a . For both models, the flutter size was also observed to increase with increase in the nonlocal parameter, e.

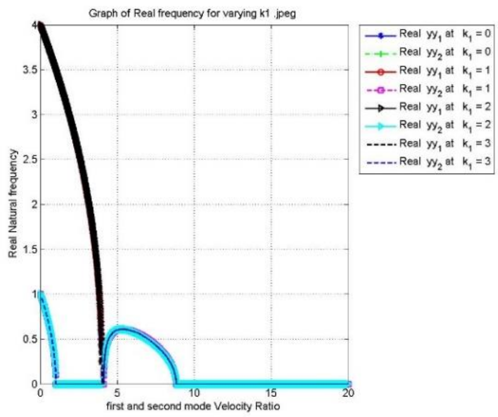

Figure 3(a): Effect of Pasternak constant on real frequency for Modified Tai-Ping Chang model

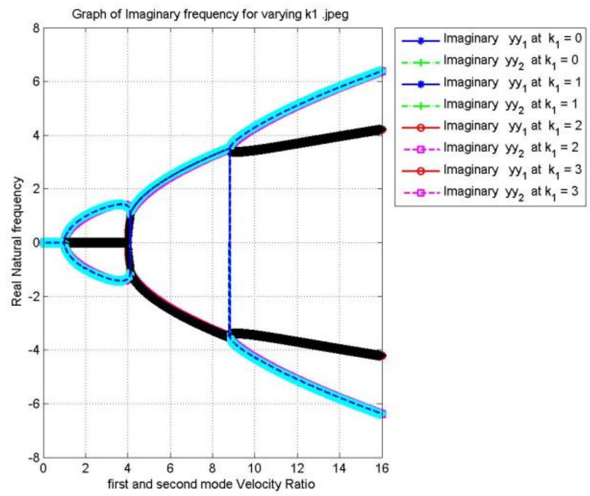

Figure 3(b): Effect of Pasternak constant on imaginary frequency for modified Tai-Ping Chang model

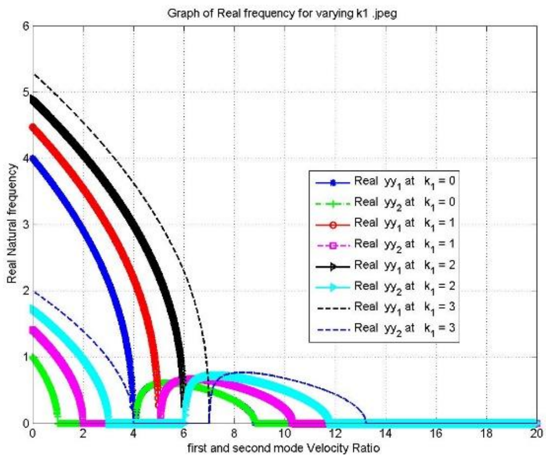

Figure 4(a): Effect of Pasternak constant on real frequency for Modified Haw-Long Lee model

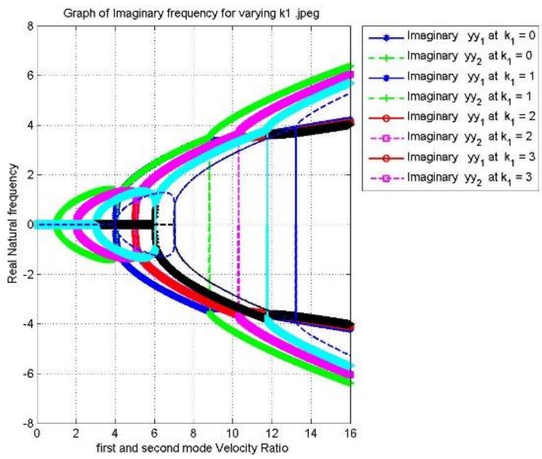

Figure 4(b): Effect of temperature on real frequency for Modified Tai-Ping Chang model

Figures 3 and 4 reveal simulation carried out at $k_2 = 0, \beta = 0.1,$ $\bar{\theta} = 0,$ $e_0 a = 0$ and varying, k_1 from 0 to 3. The Winkler foundation coefficient k_1 has no significant effect on the resonant frequencies, buckling velocities and flutter size in the modified Tai-Ping Chang *et al.* model, however, an increase in it hardens the carbon nanotube and delays buckling.

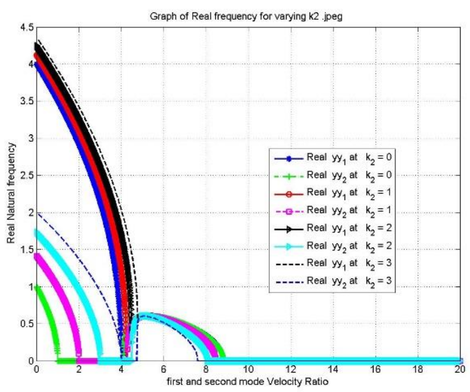

Figure 5(a): Effect of Winkler constant on real frequency for Modified Tai-Ping Chang model

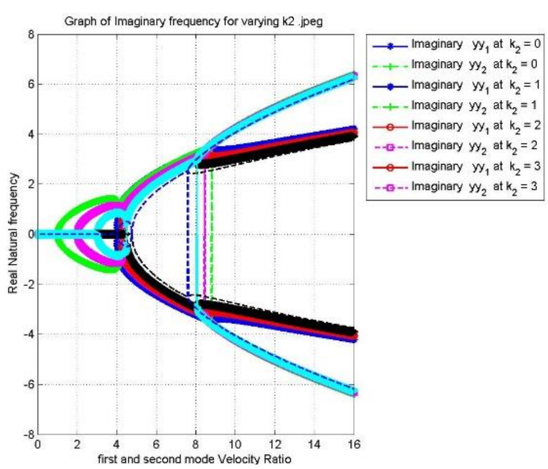

Figure 5(b): Effect of Winkler constant, k1 on imaginary frequency for modified Tai-Ping Chang model

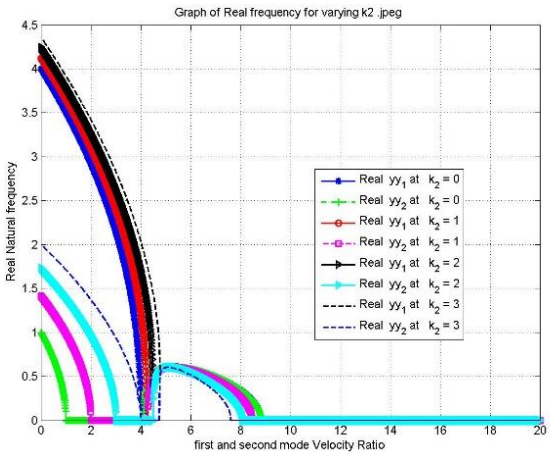

Figure 6(a): Effect of Winkler constant on real frequency for Modified Haw-Long Lee *et al*. model

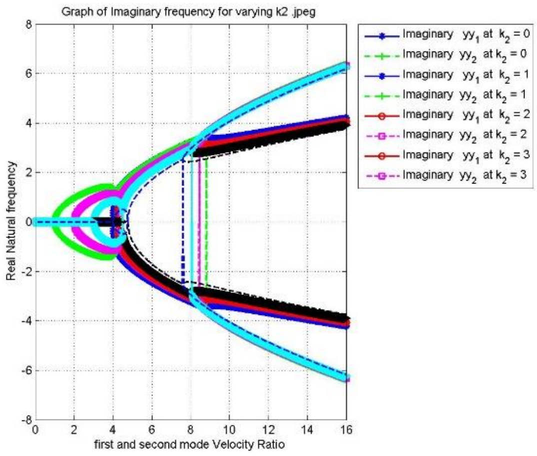

Figure 6(b): Mode shape of modified Haw-Long Lee *et al.* model for varying Winkler constant

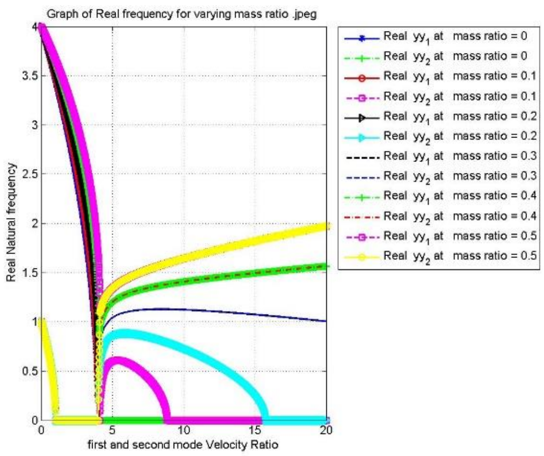

Figure 7(a): Effect of mass ratio on real frequency for Modified Tai-Ping Chang model

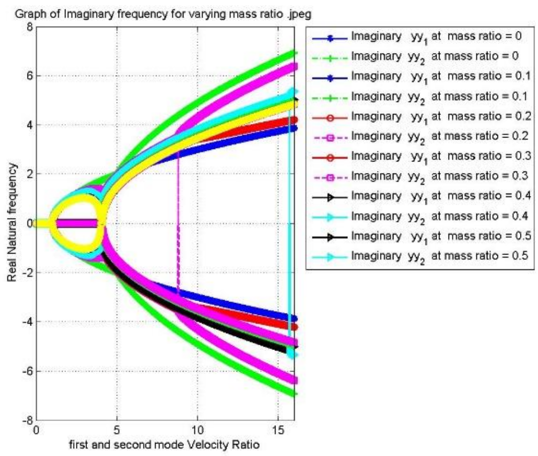

Figure 7(b): Effect of mass ratio on imaginary frequency for modified Tai-Ping Chang model

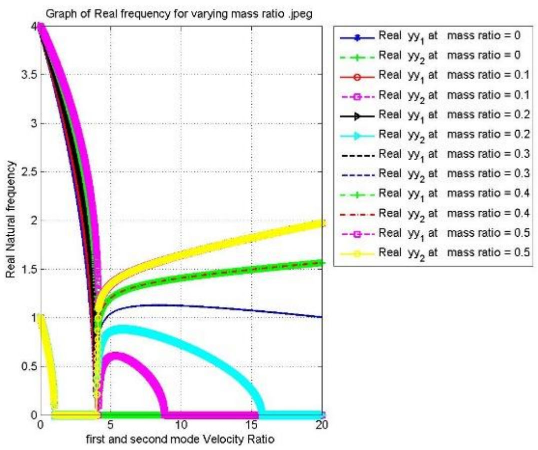

Figure 8(a): Effect of mass ratio on real frequency for Modified Haw-Long Lee model

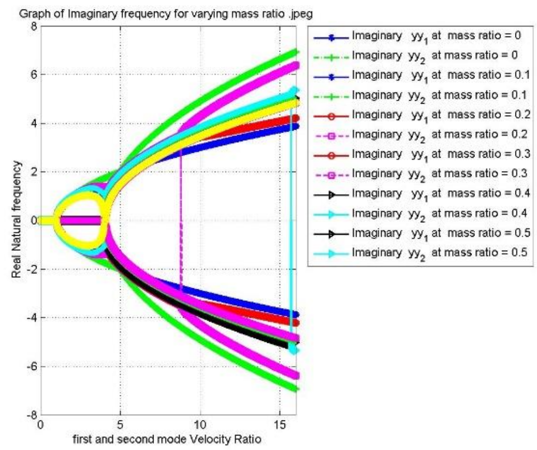

Figure 8(b) Effect of mass ratio on imaginary frequency for Modified Haw-Long Lee model

It was observed that varying values of the mass ratio has no effect on the buckling velocities. But increase in the mass ratio causes a corresponding increase in flutter size. This is depicted in **Figures 7 and 8** the effect of dimensionless mass ratio on the models. The simulation was carried out at a $k_1 = 0$, $k_2 = 0$, $N_t = 0$, $e_0 a = 0$ and β

varying from 0.1 to 0.5. In addition, for the modified Haw-Long Lee model it was observed that for values of b greater than 0.1, the second mode never buckles and for values of β greater than 0.3, the first mode flutter was incomplete while the second mode never flutter

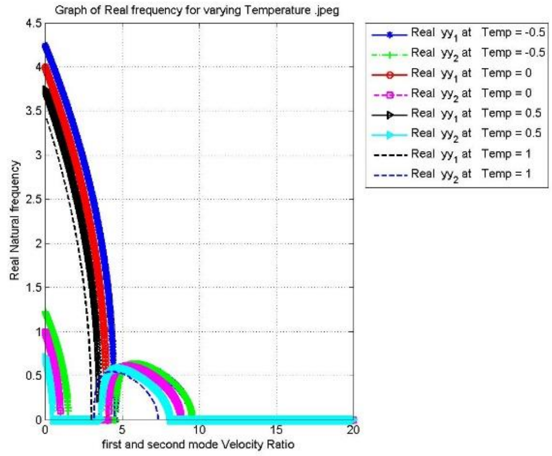

Figure 9(a): Effect of temperature on real frequency for Modified Tai-Ping Chang model

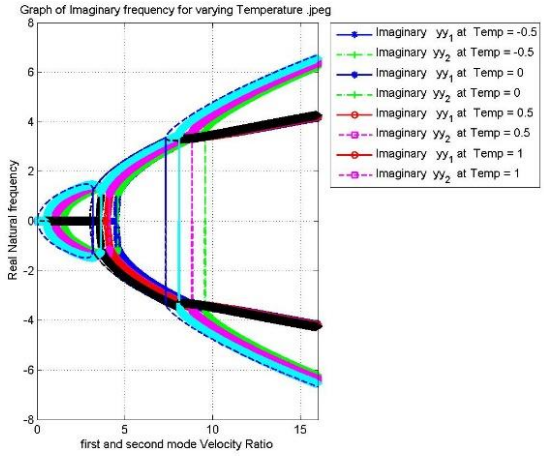

Figure 9(b): Effect of temperature on imaginary frequency for Modified Tai-Ping Chang model

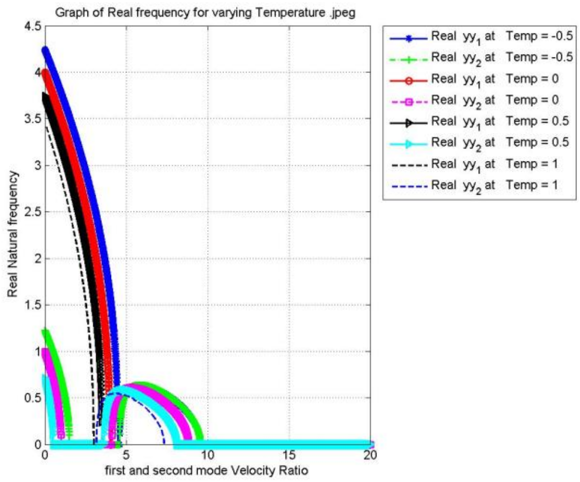

Figure 10(a): Effect of temperature on real frequency for Modified Haw-Long Lee model

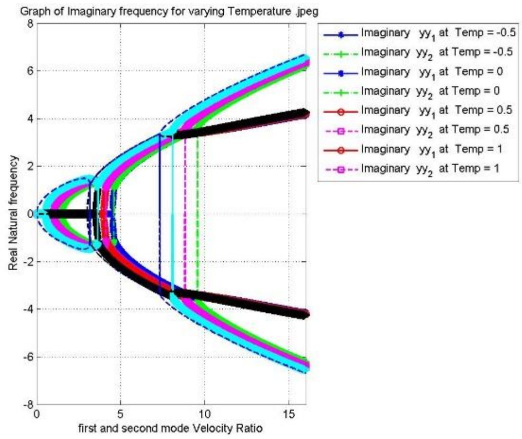

Figure 10(b): Effect of temperature on real frequency Modified Haw-Long Lee *et al* model

Increasing temperature causes reduction in first mode resonant frequency but a reverse effect on the second mode resonant frequency, as shown in **Figures 9 and 10** The simulation was carried out at $k_2 = 0$, $k_1 = 0$, $\beta = 0.1$, e_0a and varying $\bar{\theta}$ from -0.5 to 0.5. Similarly, increasing temperature decreases first mode buckling velocity while the second mode buckling velocity increased with increasing temperature. The flutter size was also observed to increase with increasing temperature.

4.0 CONCLUSION

Nonlocal Euler-Bernoulli theory with thermal effect has been applied to obtain the frequency response of a SWCNT in a two parameter elastic medium. Two models have been used to model the behaviour of the fluid conveying SWCNT interacting with both a Winkler-type and Pasternak-type elastic medium with Pinned-Pinned end conditions. The results point out the following outcomes:

(i) Increasing the temperature decreases the natural frequency but increases the second mode natural frequency of the modified Tai-Ping Chang *et al*.
(ii) The modified Haw-Long Lee *et al.* model predicts an increases in the resonant frequency as the Pasternak elastic constant, k_1, increases, but the resonant frequency of the modified Tai-ping Chang *et al* model is insensitive to changes in the Pasternak elastic constant.
(iii) Increasing the Winkler elastic constant, k_2, increases the resonant frequency for both models.
(iv) Increasing the non-local parameter, e_0a, decreases the resonant frequency for both model.
(v) The modified Tai-Ping Chang *et al.* model buckles earlier than the modified Haw-Long Lee *et al.* model.
(vi) Non-local theory predicts a larger flutter in the modified Haw-Long Lee *et al.* model compared to the modified Tai-Ping Chang *et al.* model.

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NOMENCLATURE

x = axial coordinate,
t = time,
$\mathrm{y(x,t)}$ = deflection,
V = mean flow velocity,
$\mathrm{m_f}$ = mass per unit length of fluid,
$\mathrm{m_c}$ = mass per unit length of SWCNT,
E = Young's modulus,
I = moment of inertia,
v = Poisson's ratio,
$\mathrm{k_1}$ = Pasternak elastic foundation constant,
T = Temperature change,
$\mathrm{k_2}$ = Winkler elastic foundation constant,
θ = Thermal axial force
$\propto_x$= coefficient of thermal expansion in the direction of x − axis,
$\mathrm{e_0 a}$ = non − local parameter

APPENDIX

Proof of half-range Fourier series

$$b_{np} = \frac{2}{L}\int_0^L Cos\left(\frac{n\pi x}{L}\right) Sin\left(\frac{p\pi x}{L}\right) dx$$

$$b_{np} = \frac{1}{L}\int_0^L \left\{Sin\left((p-n)\left(\frac{\pi X}{L}\right)\right) + Sin\left((p+n)\left(\frac{\pi X}{L}\right)\right)\right\} dx$$

$$b_{np} = \frac{1}{L}\left[\frac{-Cos\left((p-n)\left(\frac{\pi X}{L}\right)\right)}{\left((p-n)\left(\frac{\pi}{L}\right)\right)} - \frac{Cos\left((p+n)\left(\frac{\pi X}{L}\right)\right)}{\left((p+n)\left(\frac{\pi}{L}\right)\right)}\right]_0^L$$

$$b_{np} = \frac{-1}{\pi(p^2-n^2)}\left[(p+n)Cos\left((p-n)\left(\frac{\pi X}{L}\right)\right) + (p-n)Cos\left((p+n)\left(\frac{\pi X}{L}\right)\right)\right]$$

$$b_{np} = \frac{-1}{\pi(p^2-n^2)}\begin{cases}(p+n)Cos\big((p-n)\pi\big) + (p-n)Cos\big((p+n)\pi\big) \\ \qquad\qquad -2p\end{cases}$$

1f $(p+n)$ is odd, then $(p-n)$ is always odd and 1f $(p+n)$ is even, then $(p-n)$ is always even. Evaluating$\pm(p-n) \pm (p+n) - 2p$:

So

$$b_{np} = \begin{cases}\frac{4p}{\pi(p^2-n^2)} \\ 0\end{cases}$$

Thus, $Cos\left(\frac{n\pi X}{L}\right) = \sum_{p=1,2,3,\ldots} b_{np}\, Sin\left(\frac{n\pi X}{L}\right)$